Mono-monostatic Bodies: The Answer to Arnold's Question

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As V. I. Arnol'd conjectured; convex, homogeneous bodies with less than four equilibria (also called *mono-monostatic* bodies) may exist. Not only did his conjecture turn out to be true, the newly discovered objects show various interesting features. Our goal is to give an overview of these findings based on [7], as well as to present some new results. We will point out that mono-monostatic bodies are neither flat, nor thin, they are not similar to typical objects with more equilibria, and they are hard to approximate by polyhedra. Despite these "negative" traits, there seems to be an indication that these forms appear in Nature due to their special mechanical properties.

Do Mono-monostatic Bodies Exist?

In his recent book [11] V. I. Arnold presented a rich collection of problems sampled from his famous Moscow seminars. As Tabachnikov points out in his lively review [12], a central theme is geometrical and topological generalization of the classical Four-Vertex Theorem [2], stating that *a plane curve has at least four extrema of curvature.* The condition that some integer is *at least four* appears in numerous different problems in the book, in areas ranging from optics to mechanics. Being one of Arnold's long-term research interests, this was the central theme to his plenary lecture in 1995, Hamburg, at the International Conference on Industrial and Applied Mathematics, presented to more than 2000 mathematicians (see the accompanying article). The number of equilibria of homogeneous, rigid bodies presents a big temptation to believe in yet another emerging example of *being at least four* (in fact, the planar case was *proven* to be an example [1]). Arnold resisted and conjectured that, counter to everyday intuition and experience, the three-dimensional case might be an exception. In other words, he suggested that convex, homogeneous bodies *with fewer than four equilibria* (also called *mono-monostatic*) may exist. As often before, his conjecture proved not only to be correct but to open up an interesting avenue of mathematical thought coupled with physical and biological applications, which we explore below.

Why Are They Special?

We consider bodies resting on a horizontal surface in the presence of uniform gravity. Such bodies with just *one* stable equilibrium are called *monostatic* and they appear to be of special interest. It is easy to construct a monostatic body, such as a popular children's toy called "Comeback Kid" (Figure 1A). However, if we look for *homogeneous, convex* monostatic bodies, the task is much more difficult. In fact, in the 2D case one can prove [1] that among planar (slab-like) objects rolling along their circumference *no* monostatic bodies exist. (This statement is equivalent to the famous Four-Vertex Theorem [2] in differential geometry.)

The proof for the 2D case is indirect and runs as follows. Consider a convex, homogeneous planar "body" $\mathbb{B}$ and a polar coordinate system with origin at the center of gravity of $\mathbb{B}$. Let the continuous function $R(\varphi)$ denote the boundary of $\mathbb{B}$. As demonstrated in [1], non-degenerate stable/unstable equilibria of the body correspond to local minima/maxima of $R(\varphi)$. Assume that $R(\varphi)$ has only one local maximum and one local minimum. In this case there exists exactly one value $\varphi = \varphi_0$ for which $R(\varphi_0) = R(\varphi_0 + \pi)$; moreover, $R(\varphi) > R(\varphi_0)$ if $\pi > \varphi - \varphi_0 > 0$, and $R(\varphi) < R(\varphi_0)$ if $-\pi < \varphi\text{-}\varphi_0 < 0$ (see Figure 2A). The straight line $\varphi = \varphi_0$ (identical to $\varphi = \varphi_0 + \pi$) passing through the origin O cuts $\mathbb{B}$ into a "thin" ($R(\varphi) < R(\varphi_0)$) and a "thick" ($R(\varphi) > R(\varphi_0)$) part. This implies that *O can not be the center of gravity,* i.e., it contradicts the initial assumption.

Not surprisingly, the 3D case is more complex. Although one can construct a homogeneous, convex monostatic body (Figure 1B), the task is far less trivial if we look for a mono-

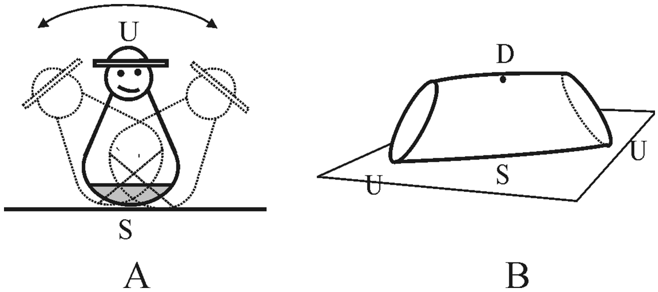

Figure 1. A. Children's toy with one stable and one unstable equilibrium inhomogeneous, mono-monostatic body), also called the "comeback kid." B. Convex, homogeneous solid body with one stable equilibrium (monostatic body). In both plots, S, D, and U denote points of the surface corresponding to stable, saddle-type, and unstable equilibria of the bodies, respectively.

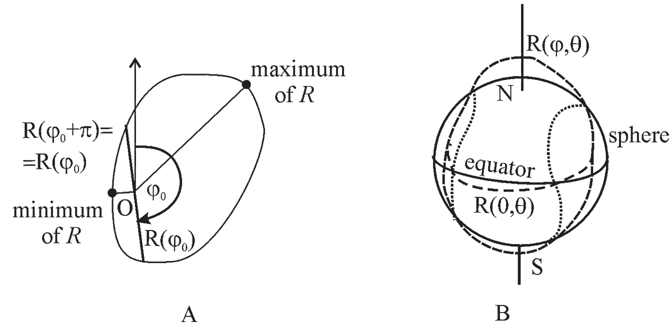

Figure 2. A. Example of a convex, homogeneous, planar body bounded by $R(\varphi)$ (polar distance from the origin O). Assuming $R(\varphi)$ has only two local extrema, the body can be cut to a "thin" and a "thick" half by the line $\varphi = \varphi_0$. Its center of gravity is on the "thick" side, in particular, it cannot coincide with O. B. 3D body (dashed line) separated into a "thin" and a "thick" part by a tennis ball-like space curve C (dotted line) along which $R = R_0$. Continuous line shows a sphere of radius R_0, which also contains the curve C.

static polyhedron with a *minimal number* of faces. Conway and Guy [3] constructed such a polyhedron with 19 faces (similar to the body in Figure 1B); it is still believed that this is the minimal number. It was shown by Heppes [6] that no homogeneous, monostatic tetrahedron exists. However, Dawson [4] showed that homogeneous, monostatic simplices exist in $d > 7$ dimensions. More recently, Dawson and Finbow [5] showed the existence of monostatic tetrahedra—but with inhomogeneous mass density.

One can construct a rather transparent classification scheme for bodies with exclusively non-degenerate balance points, based on the number and type of their equilibria. In 2D, stable and unstable equilibria always occur in pairs, so we say that a body belongs to class {1} ($i > 0$) if it has exactly $S = i$ stable (and thus, $U = i$ unstable) equilibria. As we showed above, class {1} is empty. In 3D we appeal to the Poincaré-Hopf Theorem [8], stating for convex bodies that $S + U - D = 2$, S, U, D denoting the number of local minima, maxima, and saddles of the body's potential energy; so class $\{i, j\}$ ($i, j > 0$) contains all bodies with $S = i$ stable, $U = j$ "unstable," and $D = i + j - 2$ saddle-type equilibria.

Monostatic bodies are in classes $\{1, j\}$; we will refer to the even more special class {1,1} with just one stable and one unstable equilibrium as "*mono-monostatic.*" While in 2D being monostatic implies being mono-monostatic (and vice versa), the 3D case is more complicated: a monostatic body could have, in principle, any number of unstable equilibria (e.g., the body in Figure 1B belongs to class {1,2} and has four equilibria altogether, as pointed out by Arnold, see story). *Arnold's conjecture was that class* {1,1} *is not empty*, i.e., *that homogeneous, convex mono-monostatic bodies existed.* Before we outline the construction of such an object, we want to highlight its very special relation to other convex bodies.

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Intuitively it seems clear that by applying small, local perturbations to a surface, one may produce additional local maxima and minima (close to existing ones), similar to the "egg of Columbus." According to some accounts, Christopher Columbus attended a dinner which a Spanish gentleman had given in his honor. Columbus asked the gentlemen in attendance to make an egg stand on one end. After the gentlemen successively tried to and failed, they stated that it was impossible. Columbus then placed the egg's small end on the table, breaking the shell a bit, so that it could stand upright. Columbus then stated that it was "the simplest thing in the world. Anybody can do it, after he has been shown how!" In [7] we showed that in an analogous manner, one can *add* stable and unstable equilibria *one by one* by taking away locally small portions of the body. Apparently, the inverse is not possible, i.e., for a *typical* body one cannot decrease the number of equilibria via small perturbations.

This result indicates the special status of mono-monostatic bodies among other objects. For any given typical mono-monostatic body, one can find bodies in an arbitrary class $\{i, j\}$ which have *almost the same shape*. On the other hand, to any typical member of class $\{i, j\}$, $(i, j > 1)$, one can *not* find a mono-monostatic body which has almost the same shape. This may explain why mono-monostatic bodies do not occur often in Nature, also, why it is difficult to visualize such a shape. Next we will demonstrate such an object.

What Are They Like?

As in the planar case, a mono-monostatic 3D body can be cut to a "thin" and a "thick" part by a closed curve on its boundary, along which $R(\theta,\varphi)$ is constant. If this separatrix curve happens to be planar, its existence leads to contradiction, similar to the 2D case. (If, for example, it is the "equator" $\varphi = 0$ and $\varphi > 0/\varphi < 0$ are the thick/thin halves, the center of gravity should be on the upper ($\varphi > 0$) side of the origin). However, in case of a generic spatial separatrix, the above argument no longer applies. In particular, the curve can be similar to the ones on the surfaces of tennis balls (Figure 2B). In this case the "upper" thick ("lower" thin) part is partially below (above) the equator; thus it is possible to have the center of gravity at the origin. Our construction will be of this type. We define a suitable two-parameter family of surfaces $R(\theta,\varphi,c,d)$ in the spherical coordinate system (r,θ,φ) with $-\pi/2 < \varphi < \pi/2$ and $0 \le \theta \le 2\pi$, or $\varphi = \pm\,\pi/2$ and no θ coordinate, while $c > 0$ and $0 < d < 1$ are parameters. Conveniently, R can be decomposed in the following way:

$$R(\theta,\varphi,c,d) = (1 + d)\cdot \Delta R(\theta,\varphi,c), \qquad (1)$$

where ΔR denotes the *type* of deviation from the unit sphere. "Thin"/"thick" parts of the body are characterized by negativeness/positiveness of ΔR (i.e., the separatrix between the thick and thin portions will be given by $\Delta R = 0$), while the parameter d is a measure of how far the surface is from the sphere. We will choose small values of d so as to make the surface convex. Now we proceed to define ΔR.

We will have the maximum/minimum points of ΔR ($\Delta R = \pm 1$) at the North/South Pole ($\varphi = \pm\pi/2$). The shapes of the thick and thin portions of the body are controlled by the parameter c: for $c >> 1$ the separatrix will approach the equator; for smaller values of c, the separatrix will become similar to the curve on the tennis ball.

Consider the following smooth, one-parameter mapping $f(\varphi,c)$: $(-\pi/2,\pi/2) \to (-\pi/2,\pi/2)$:

$$f(\varphi,c) = \pi \cdot \left[\frac{e^{\left[\frac{\varphi}{\pi c} + \frac{1}{2c}\right]} - 1}{e^{1/c} - 1} - \frac{1}{2}\right]. \qquad (2)$$

For large values of the parameter ($c >> 1$), this mapping is almost the identity; however, if c is close to 0, there is a large deviation from linearity. Based on (2), we define the related maps

$$f_1(\varphi,c) = \sin(f(\varphi,c)) \qquad (3)$$

and

$$f_2(\varphi,c) = -f_1(-\varphi,c). \qquad (4)$$

We will choose ΔR so as to obtain $\Delta R(\varphi,\theta,c) = f_2(\varphi,c)$ if $\theta = \pi/2$ or $3\pi/2$ (i.e., a big portion of these sections of the body lie in the thick part, cf. Figure 2B) and $\Delta R = f_2$ if $\theta = 0$ or π (the majority of these sections are in the thin part). The function

$$a(\theta,\varphi,c) = \frac{\cos^2(\theta)\cdot(1 - f_1{}^2)}{\cos^2(\theta)(1 - f_1{}^2) + \sin^2(\theta)\cdot(1 - f_2{}^2)} = \qquad (5)$$

where $|\varphi| < {}^{\pi}\!/_2$ QU1

$$= \frac{1}{1 + \tan^2(\theta)\dfrac{\cos^2(f(\varphi,c))}{\cos^2(f(\varphi,c))}}$$

is used to construct ΔR as a weighted average of f_1 and f_2 in the following way:

$$\Delta R(\theta,\varphi,c) = \left\{\begin{array}{ll} a\cdot f_1 + (1 - a)\cdot f_2 & \text{if } |\varphi| < {}^{\pi}\!/_2 \\ 1 & \text{if } \varphi = {}^{\pi}\!/_2 \\ -1 & \text{if } \varphi = -{}^{\pi}\!/_2 \end{array}\right\}. \qquad (6)$$

The choice of the function a guarantees, on the one hand, the gradual transition from f_1 to f_2 if θ is varied between 0 and $\pi/2$. On the other hand, it was chosen to result in the desired shape of thick/thin halves of the body (Figure 2/B). The function R defined by equations (1)–(6) is illustrated in Figure 3 for intermediate values of c and d. For $c >>$ 1, the constructed surface $R = 1 + d\Delta R$ is separated by the $\varphi = 0$ equator into two unequal halves: the upper ($\varphi > 0$) half is "thick" ($R > 1$) and the lower ($\varphi < 0$) half is "thin" ($R < 1$). By decreasing c, the line separating the "thick" and "thin" portions becomes a space curve; thus the thicker portion moves downward and the thinner portion upward. As c approaches zero, the upper half of the body becomes thin and the lower one becomes thick (cf. Figure 4.)

F3 F4

In [7] we proved analytically that there exist ranges for c and d where the body is convex and the center of gravity is at the origin, i.e. it belongs to class {1.1}. Numerical studies suggest that d must be very small ($d < 5 \cdot 10^{-5}$) to satisfy convexity together with the other restrictions, so the created object is very similar to a sphere. (In the admitted range of d, the other parameter is approximately $c \approx 0.275$.)

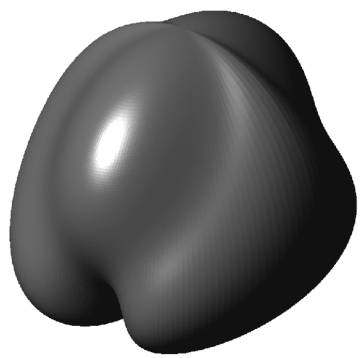

Figure 3. Plot of the body if $c = d = 1/2$

What Are They *Not* Like?

Intuitively, it appears that mono-monostatic bodies can be neither very *flat* nor very *thin*; the former shape would have at least two stable equilibria; the latter, at least two unstable equilibria. To make this intuition more exact, we define the flatness F and thinness T of a body. Draw a closed curve c on the surface, traced by the position vector $R(s)$, $s \in [0,1]$ from the center of gravity O. Pick two points P_i ($i = 1, 2$) on *opposite* sides of c, with position vectors R_i ($i = 1, 2$), respectively. We define the flatness and thinness as

$$F = \sup_{\forall c, P_1, P_2} \left\{ \frac{\min_s(R(s))}{\max_i(R_i)} \right\}, \quad T = \sup_{\forall c, P_1, P_2} \left\{ \frac{\min_i(R_i)}{\max_s(R(s))} \right\}.$$

Although F and T are hard to compute for a general case, it is easy to give both a problem-specific and a general lower bound. For the latter, we have

$$F, T \geq 1, \tag{7}$$

since $F = T = 1$ can be always obtained by shrinking the curve c to a single point. For "simple" objects F and T can be determined, and the values agree fairly well with intuition in Table 1.

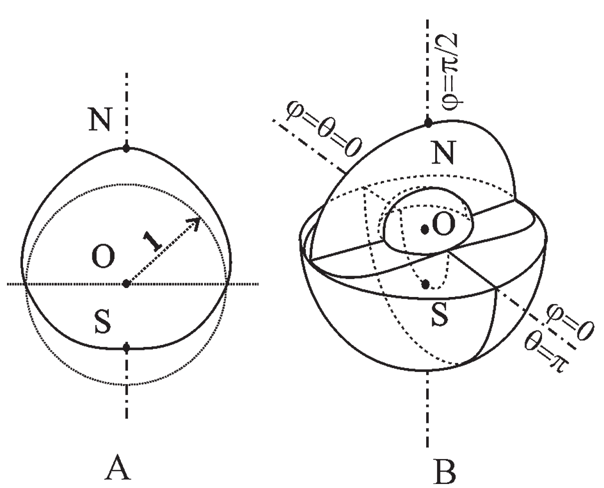

Figure 4. A. Side view of the body if $c >> 1$ (and $d \approx 1/3$). Note that $\Delta R > 0$ if $\varphi > 0$ and $\Delta R < 0$ if $\varphi < 0$. B. Spatial view if $c << 1$. Here, $\Delta R > 0$ typically for $\varphi < 0$ and vice versa.

Table 1. The flatness and thinness of some "simple" objects

Body	Flatness F	Thinness T
Sphere	1	1
Regular tetrahedron	$\sqrt{3}$	$\sqrt{3}$
Cube	$\sqrt{2}$	$\sqrt{(3/2)}$
Octahedron	$\sqrt{(3/2)}$	$\sqrt{2}$
Cylinder with radius r, height 2 h, $z = \sqrt{(r^2 + h^2)}$	z/h	z/r
Ellipsoid with axes $a < b < c$	b/a	c/b

Now we show that F and T are related to the number S of stable and U of unstable equilibria by

Lemma 1: *(a)* $F = 1$ *if and only if* $S = 1$ *and (b)* $T = 1$ *if and only if* $U = 1$.

We only prove (a); the proof of (b) runs analogously.

If $S > 1$, then there exists one global minimum for the radius R and at least one additional (local) minimum. Select c as a closed, $R = R_0 =$ *constant* curve, circling the local minimum very closely. Select the points P_1 and P_2 coinciding with global and local minima, respectively. Now we have $R_1 \leq R_2 < R_0$ and $min(R(s)) = R_0$, $max(R_i) = R_2$, so $S > 1$ implies $F > 1$.

If $S = 1$, then R has only one minimum, so it assumes only values greater than or equal to $min(R(s))$ on one side of the curve c, so $F \leq 1$, but due to (7), we have $F = 1$. Q.e.d.

Lemma 1 confirms our initial intuition that mono-monostatic bodies can be *neither flat, nor thin.* In fact, they have simultaneously minimal flatness and minimal thinness; moreover, they are the only non-degenerate bodies having this property.

Another interesting though somewhat "negative" feature of mono-monostatic bodies is the apparent lack of any simple polyhedral approximation. As mentioned before, the existence of *monostatic* polyhedra with minimal number of faces has been investigated [3],[4],[5],[6]. One may generalize this to the existence of polyhedra *in class* $\{i, j\}$, with minimal number of faces. Intuitively it appears evident that polyhedra exist in each class: if we construct a sufficiently fine triangulation on the surface of a smooth body in class $\{i, j\}$ with vertices at unstable equilibria, edges at saddles and faces at stable equilibria; then the resulting polyhedron may—at sufficiently high mesh density and appropriate mesh ratios—"inherit" the class of the approximated smooth body. It also appears that if the topological inequalities $2i \geq j + 4$ and $2j \geq i + 4$ are valid, then we can have "minimal" polyhedra, where the number of stable equilibria equals the number of faces, the number of unstable equilibria equals the number of vertices, and the number of saddles equals the number of edges. Much more puzzling appear to be the polyhedra in classes *not satisfying* the above topological inequalities: a special case of these polyhedra are monostatic ones; however, many other types belong here as well. In particular, it would be of interest to know the minimal number of faces of a polyhedron in class {1,1}. We can imagine such a polyhedron as an approximation of a smooth mono-monostatic body. Since the latter are close

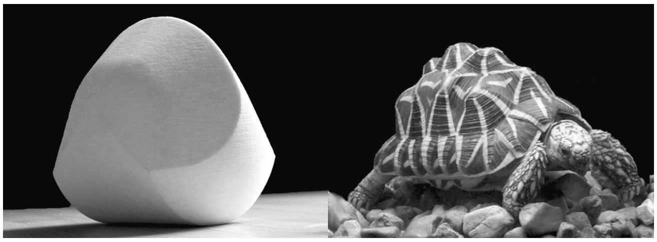

Figure 5. Mono-monostatic body and Indian Star Tortoise (*Geochelone elegans*).

to the sphere (they are neither flat nor thin), the number of equilibria is particularly sensitive to perturbations, so the minimal number of faces of a mono-monostatic polyhedron may be a very large number.

Mono-monostatic Bodies *do* Exist

Arnold's conjecture proved to be correct: there exist homogeneous, convex bodies with just two equilibria; we called these objects mono-monostatic.

Based on the results presented so far, one must get the impression that mono-monostatic bodies are *hiding*—that they are hard to visualize, hard to describe, and hard to identify. In particular, we showed that their form is not similar to any typical representative of any other equilibrium class. We also showed that they are *neither flat, nor thin*; in fact, they are the only non-degenerate objects having simultaneously minimal flatness and thinness. Imagining their polyhedral approximation seems to be a futile effort as well: the minimal number of faces for mono-monostatic polyhedra might be very large. The extreme physical fragility of these forms (i.e., their sensitivity to local perturbations due to abrasion) was also confirmed by statistical experiments on pebbles (reported in [7]); in a sample of 2000 pebbles not a single mono-monostatic object could be identified. Apparently, mono-monostatic bodies escape everyday human intuition.

They did not escape Arnold's intuition. Neither does Nature ignore these mysterious objects: being monostatic can be a life-saving property for land animals with a hard shell, such as beetles and turtles. In fact, the "righting response" (their ability to turn back when placed upside down) of these animals is often regarded as a measure of their fitness ([9],[10]). Although the example presented above under "Why Are They Special," proved to be practically indistinguishable from the sphere, rather different forms are also included in the mono-monostatic class. In particular, we identified one of these forms, which not only shows substantial deviation from the sphere, but also displays remarkable similarity to some turtles and beetles. We built the object by using 3D printing technology, and in Figure 5 it can be visually compared to an Indian Star Tortoise (*Geochelone elegans*). F5

Needless to say, the analogy is incomplete: turtles are neither homogeneous nor mono-monostatic. (They do not need to be exactly mono-monostatic; righting is assisted dynamically by the motion of the limbs.) On the other hand, being that close to a mono-monostatic form is probably not just a coincidence; as we indicated before, such forms are unlikely to be found by chance, either by us or by Evolution itself.

ACKNOWLEDGEMENT

The support of OTKA grant TS49885 is gratefully acknowledged.

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QU1
Give position of line starting with “where.”